Cosmological Numerical Relativity

New Perspectives on gravitation & cosmology

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Motivation

- There exists a scale on which universe can be considered homogeneous. ($\sim 80h^{-1}Mpc$)
- However it is not homogeneous and isotropic on smaller scale.
- Cosmological Perturbation : Background + Perturbation
    - Non-linear regime will emerge.
    - Exteremly difficult!
- Cosmological N-body : evolution of background from averaged density
    - In general, evolution of mean ≠ mean of evolution
- The validity of these must be checked against a more precise solution.
- Numerical Relativity : Full GR simulation is possible
- Upcoming cosmological survey : Euclid, SKA and LSST

Contents

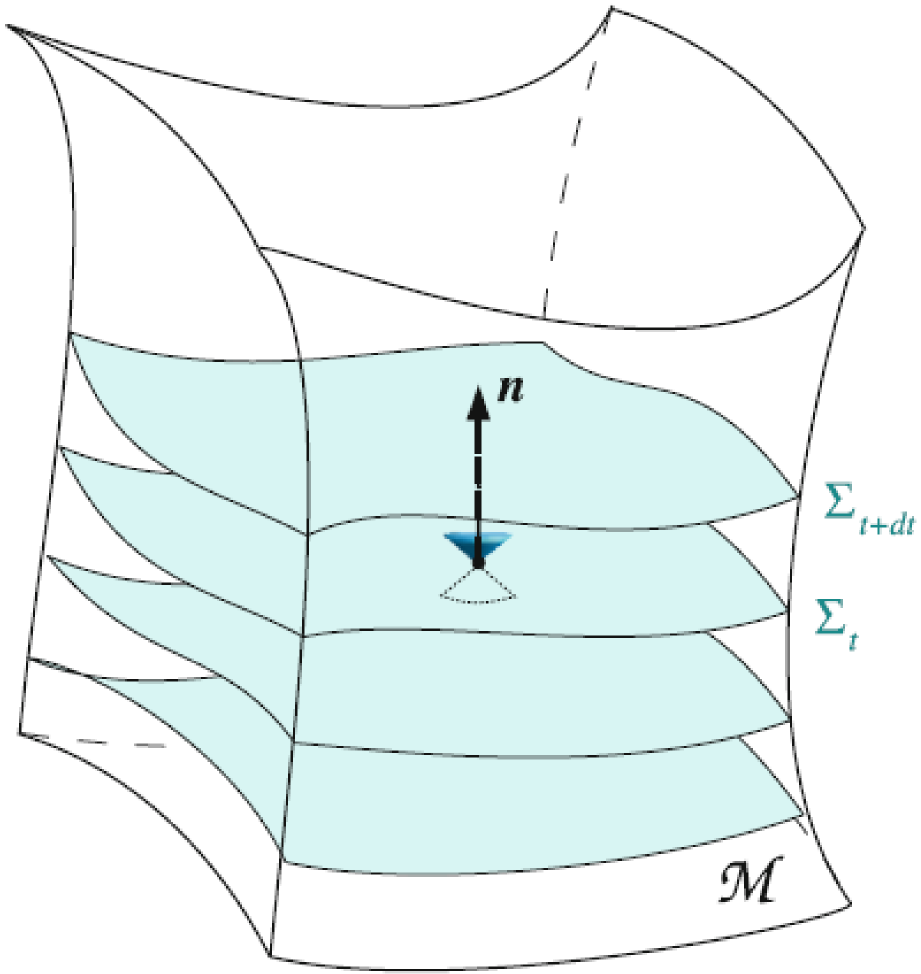
3+1 Formalism

Geometrodynamics Point of View

- Time evolution of 3-dimensional space
- Questions
  - Is there global time function?
  - Initial value problem is possible?
  - How to decompose Einstein equation?
    - Evolution equation : time derivative
    - Constraint equation : spatial derivative
  - Gauge freedoms of time function?

Globally Hyperbolic Spacetime

- There exist global time function t
  - Σ_t : constant t hypersurfaces
  - Spacetime is foliated by Σ_t.
  - Σ_t is spacelike Cauchy surface.



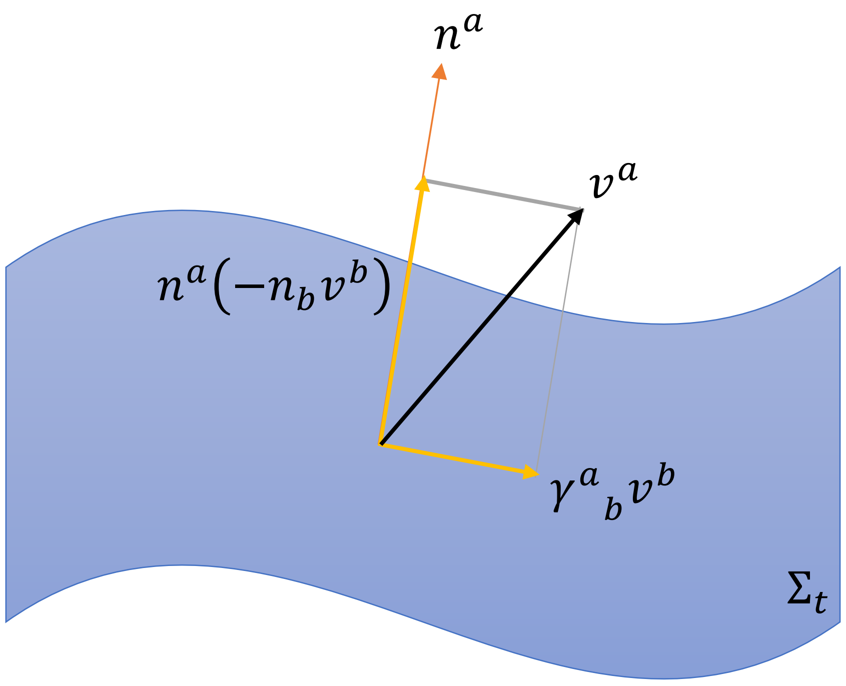
Initial Value Problem (vacuum case)

- Given vacuum initial data, we can always **uniquely develop** a spacetime satisfies the vacuum Einstein equation. [Y. Choquet-Bruhat and R. Geroch (1969)]
- Initial value problem is also possible in perfect fluid case. [S. Hawking and G. Ellis (1973)]

Y. Choquet-Bruhat

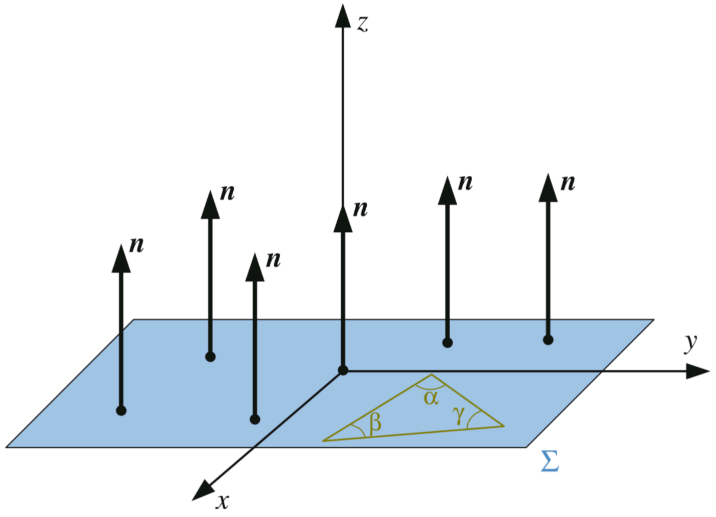
Unit Normal Vector and 3+1 Decomposition

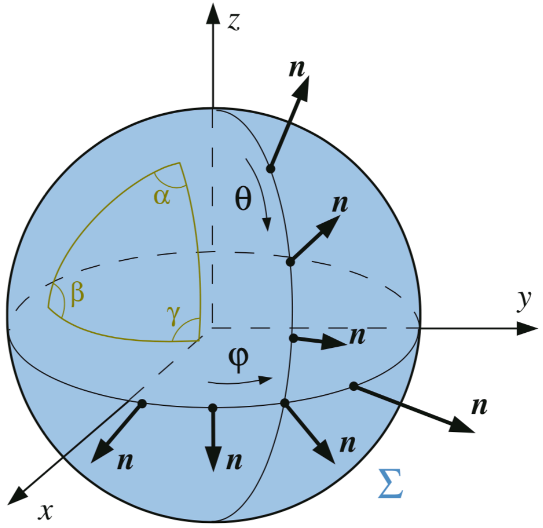
- $(dt)_a$: gradient of t
- $n^a = \frac{-g^{ab}(dt)_b}{\sqrt{-g^{cd}(dt)_c(dt)_d}}$: future-directed unit normal to Σ_t
- $N \equiv \frac{1}{\sqrt{-g^{cd}(dt)_c(dt)_d}}$: Lapse function
- Projection to Σ_t
  - $v^a + n^a(n_b v^b) = [\delta^a{}_b + n^a n_b]v^b$
  - $\gamma^a{}_b \equiv \delta^a{}_b + n^a n_b$
- 3+1 Decomposition of vector
  - $v^a = (-v^b n_b)n^a + \gamma^a{}_b v^b$



Fundamental Forms of Σ_t

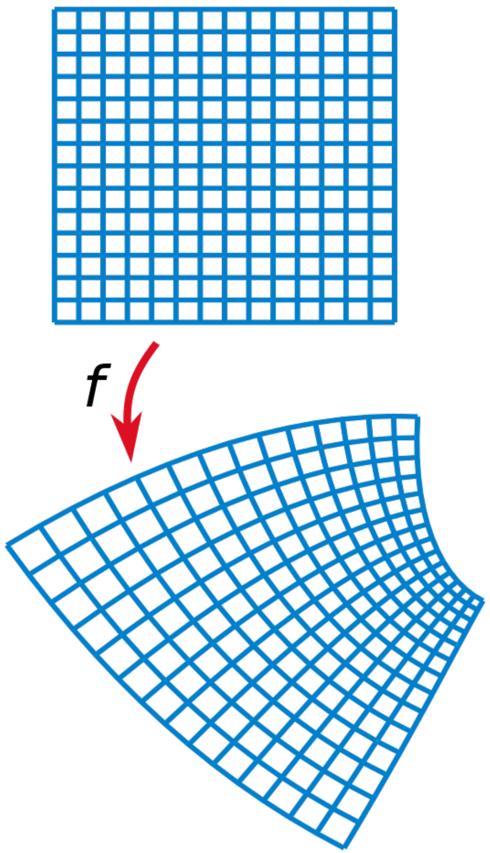
- 1st fundamental form : induced metric
  - $\gamma_{ab} \equiv g_{cd}\gamma^c{}_a\gamma^d{}_b$
  - Metric for spatial vector.
- 2nd fundamental form : extrinsic curvature
  - $K_{ab} \equiv \frac{1}{2}\mathcal{L}_n\gamma_{ab} = \gamma^c{}_a\nabla_c n_b$
  - "Time derivative" of induced metric.
  - Measure of how to embed Σ_t in spacetime.
- Associated Derivative
  - $D_c\gamma_{ab} = 0$





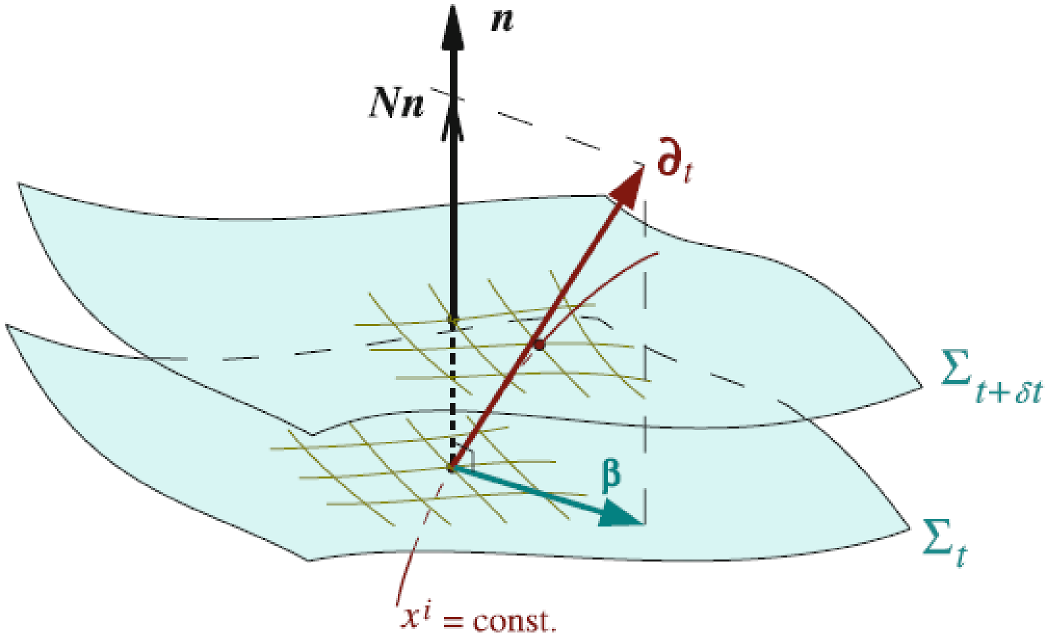
Conformal Decomposition

- $\gamma_{ab} = \Psi^4 \tilde{\gamma}_{ab}$
- $\widetilde{D}_c \tilde{\gamma}_{ab} = 0$
- $K_{ab} = \frac{1}{3} K \gamma_{ab} + A_{ab}$
  $= \frac{1}{3} K \gamma_{ab} + \Psi^4 \tilde{A}_{ab}$

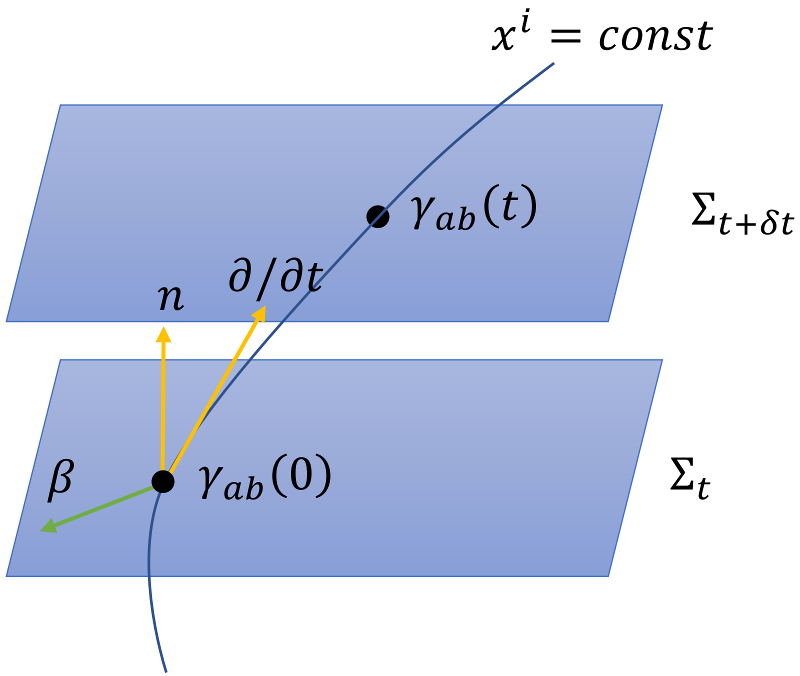


Gauge in Geometrodynamics

- $(\partial/\partial t)^a = Nn^a + \beta^a$
- β^a : shift
  - spatial gauge
  - It determines propagation direction of spatial coordinate
- N : lapse
  - time slicing
  - It determines shape of next time slice.



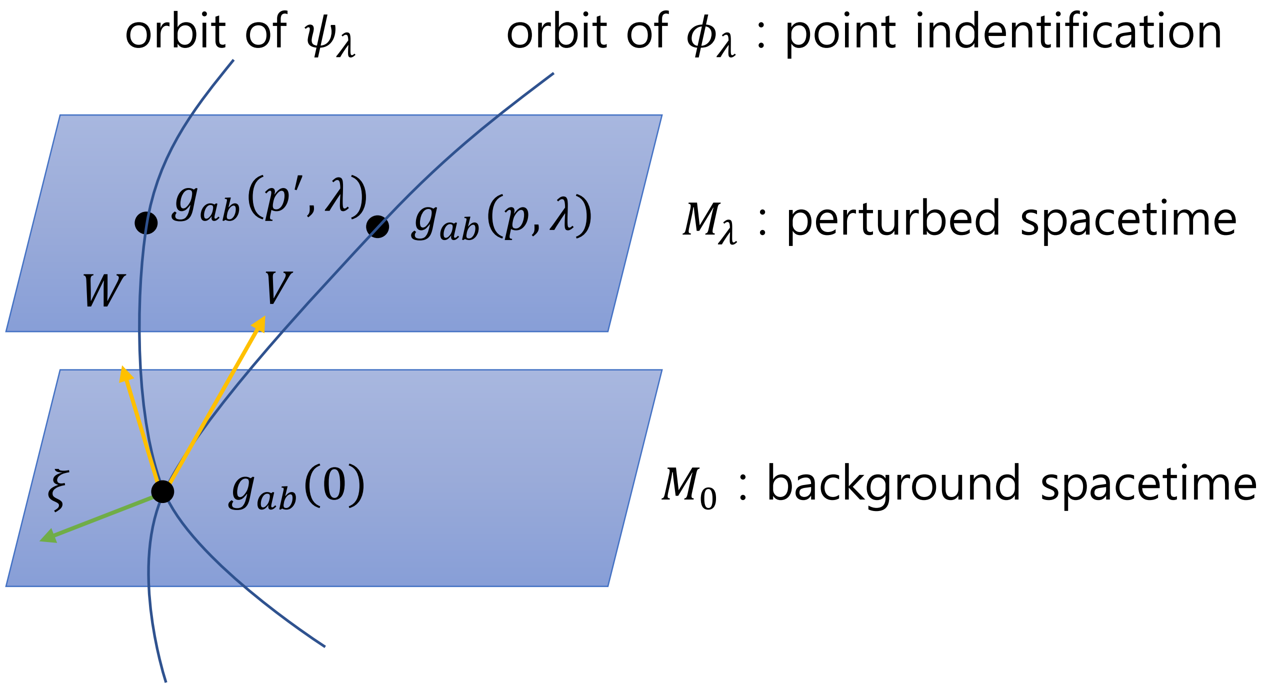
It is different to Perturbation Gauge



$\partial/\partial t$: Spatial Gauge

$\mathcal{L}_{(\partial/\partial t)}\gamma_{ab} - \mathcal{L}_n\gamma_{ab} = \mathcal{L}_\beta\gamma_{ab}$
$= D_a\beta_b + D_b\beta_a$

[Gauge in geometrodynamics]



V : Perturbation Gauge

$\mathcal{L}_W g_{ab} - \mathcal{L}_V g_{ab} = \mathcal{L}_\xi g_{ab}$
$= \nabla_a\xi_b + \nabla_b\xi_a$

[Gauge in perturbation of spacetime]

Example: FLRW Metric

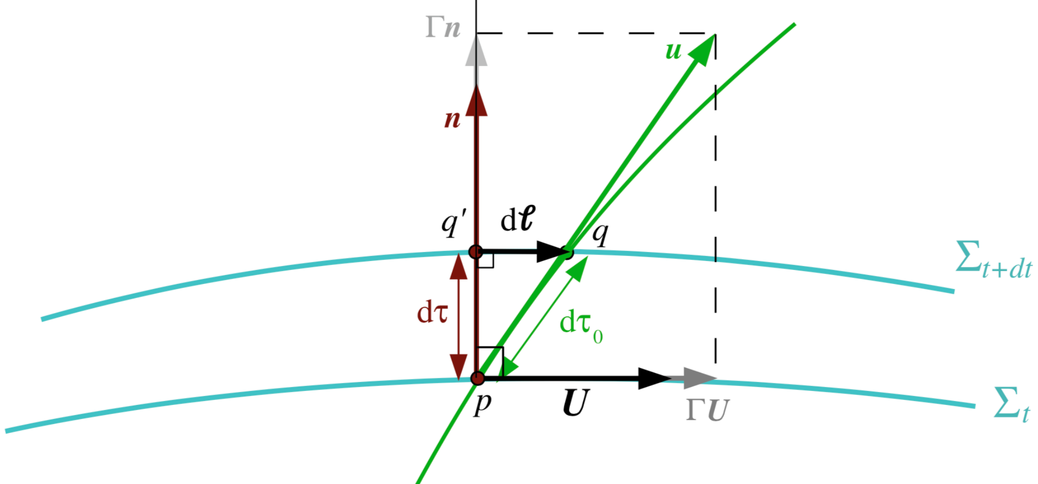
- $ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + f_k^2(\chi)d\Omega^2\right)$
  - $f_{+1}(\chi) = \sin\chi$, $f_0(\chi) = \chi$, $f_{-1}(\chi) = \sinh\chi$
- $N = 1 \qquad \beta^a = 0$
- $\Psi^2 = a \qquad$: scale factor
- $K_{ab} = a\dot{a}\tilde{\gamma}_{ab}$
  - $K = 3\frac{\dot{a}}{a}$: Hubble parameter
  - $A_{ab} = 0$
- $\tilde{R}_{abcd} = k(\tilde{\gamma}_{ac}\tilde{\gamma}_{bd} - \tilde{\gamma}_{ad}\tilde{\gamma}_{bc}) \qquad \tilde{R}_{ab} = 2k\tilde{\gamma}_{ab} \qquad \tilde{R} = 6k$

3+1 Decomposition of Stress-Energy

- $T_{ab} = En_a n_b + p_a n_b + n_a p_b + S_{ab}$
- $\nabla^b T_{ab} = 0$: Conservation
  - $\mathcal{L}_n \rho = -D_a p^a - KE - K_{ab} S^{ab} - 2p^a a_a$
  - $\mathcal{L}_n p_a = -D^b S_{ab} - S_{ab} a^b - K p_a - E a_a$

Example: Perfect Fluid

- $T_{ab} = \rho u_a u_b + (g_{ab} + u_a u_b)P$
- $u^a = \Gamma(n^a + U^a)$: 3+1 decomposition of u^a
- $\mathcal{L}_n \rho = -D_a[(\rho + P)U^a] - N^{-1}(\rho + P)\left(K + K_{ab}U^a U^b\right) - 2(\rho + P)U^a a_a$
- $\mathcal{L}_n U_a = -U^b D_b U_a + U^b a_b U_a - a_a + K_{bc}U^b U^c U_a - \frac{1}{\rho+P}[D_a P + U_a N^{-1}\mathcal{L}_n P]$



3+1 Decomposition of Einstein Equation

- $G_{ab} = 8\pi T_{ab}$
- Evolution Equations
  - $\mathcal{L}_n \ln \Psi = \frac{1}{6} K - \frac{1}{12} \mathcal{L}_n \ln f$
  - $\mathcal{L}_n \tilde{\gamma}_{ij} = 2\tilde{A}_{ij} + \frac{1}{3} \tilde{\gamma}_{ij} \mathcal{L}_n \ln f$
  - $\mathcal{L}_n K = \Psi^{-4} \left[N^{-1} \widetilde{D}_i \widetilde{D}^i N + 2\widetilde{D}_i \ln \Psi \, \widetilde{D}^i \ln N \right] - \tilde{A}_{ij} \tilde{A}^{ij} - \frac{1}{3} K^2 - 4\pi(E + S)$
  - $\mathcal{L}_n \tilde{A}_{ij} = \frac{1}{3} \tilde{A}_{ij} \mathcal{L}_n \ln f + K\tilde{A}_{ij} + 2\tilde{\gamma}^{kl} \tilde{A}_{ik} \tilde{A}_{jl} + 8\pi \left(\Psi^{-4} S_{ij} - \frac{1}{3} S \tilde{\gamma}_{ij} \right) - \Psi^{-4} \left[-N^{-1} \widetilde{D}_i \widetilde{D}_j N + 4\widetilde{D}_{(i} \ln \Psi \, \widetilde{D}_{j)} \ln N + \frac{1}{3} \left(N^{-1} \widetilde{D}_k \widetilde{D}^k N - 4\widetilde{D}_k \ln \Psi \, \widetilde{D}^k \ln N \right) \tilde{\gamma}_{ij} + \tilde{R}_{ij} - \frac{1}{3} \tilde{R} \tilde{\gamma}_{ij} - 2\widetilde{D}_i \widetilde{D}_j \ln \Psi + 4\widetilde{D}_i \ln \Psi \, \widetilde{D}_j \ln \Psi + \frac{2}{3} \left(\widetilde{D}_k \widetilde{D}^k \ln \Psi - 2\widetilde{D}_k \ln \Psi \, \widetilde{D}^k \ln \Psi \right) \tilde{\gamma}_{ij} \right]$
- Constraint equations
  - $\widetilde{D}_i \widetilde{D}^i \Psi - \frac{1}{8} \tilde{R} \Psi + \frac{1}{8} \tilde{A}_{ij} \tilde{A}^{ij} \Psi^5 - \frac{1}{12} K^2 \Psi^5 = -2\pi E \Psi^5$ [Hamiltonian Constraint]
  - $\widetilde{D}_j \left(\Psi^6 \tilde{A}^{ij} \right) - \frac{2}{3} \Psi^6 \widetilde{D}^i K = 8\pi \Psi^{10} p^i$ [Momentum Constraint]

Example: FLRW Model

- Evolution Equation
  - $K = 3\frac{\dot{a}}{a}$
  - $A_{ab} = 0$
  - $3\frac{\ddot{a}}{a} = -4\pi(\rho + 3P)$
  - $u^a = n^a$
- Constraint Equation
  - $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}$

Full GR Evolution of Cosmological Inhomogeneities

First Full GR Evolution

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Departures from the Friedmann-Lemaitre-Robertston-Walker Cosmological Model in an Inhomogeneous Universe: A Numerical Examination

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While the use of numerical general relativity for modeling astrophysical phenomena and compact objects is commonplace, the application to cosmological scenarios is only just beginning. Here, we examine the expansion of a spacetime using the Baumgarte-Shapiro-Shibata-Nakamura formalism of numerical relativity in synchronous gauge. This work represents the first numerical cosmological study that is fully relativistic, nonlinear, and without symmetry. The universe that emerges exhibits an average Friedmann-Lemaître-Robertson-Walker (FLRW) behavior; however, this universe also exhibits locally inhomogeneous expansion beyond that expected in linear perturbation theory around a FLRW background.

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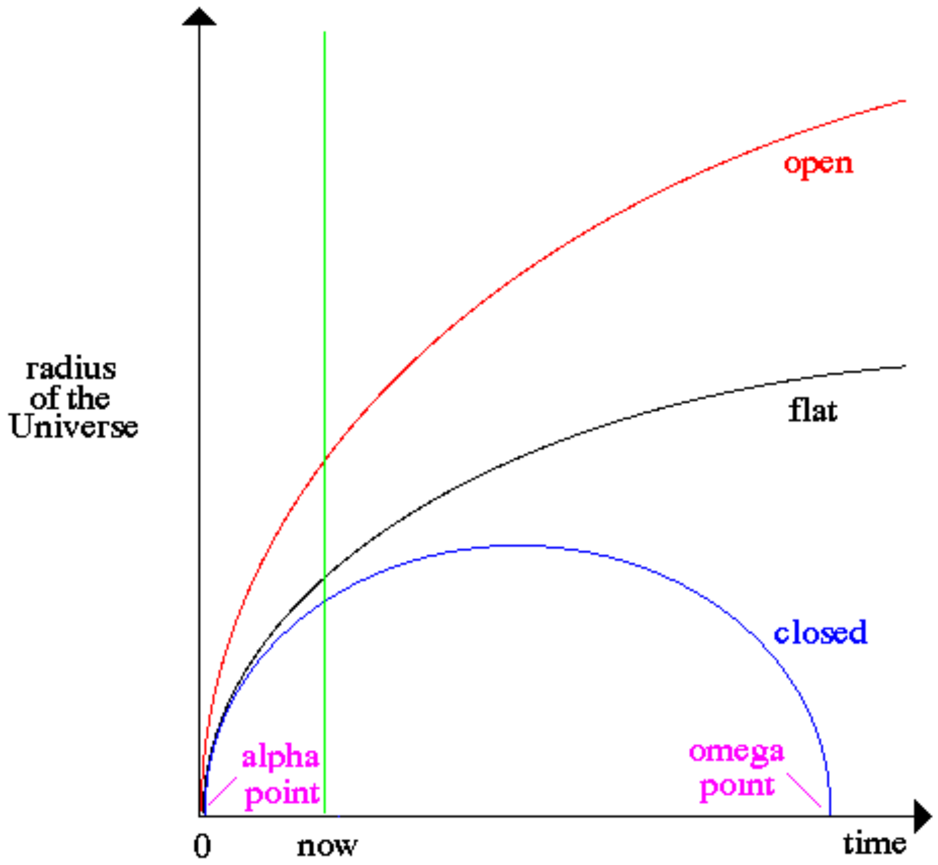
FLRW model to compare

- Dust filled flat universe
  - Dust : $P = 0$
  - Flat : $k = 0$

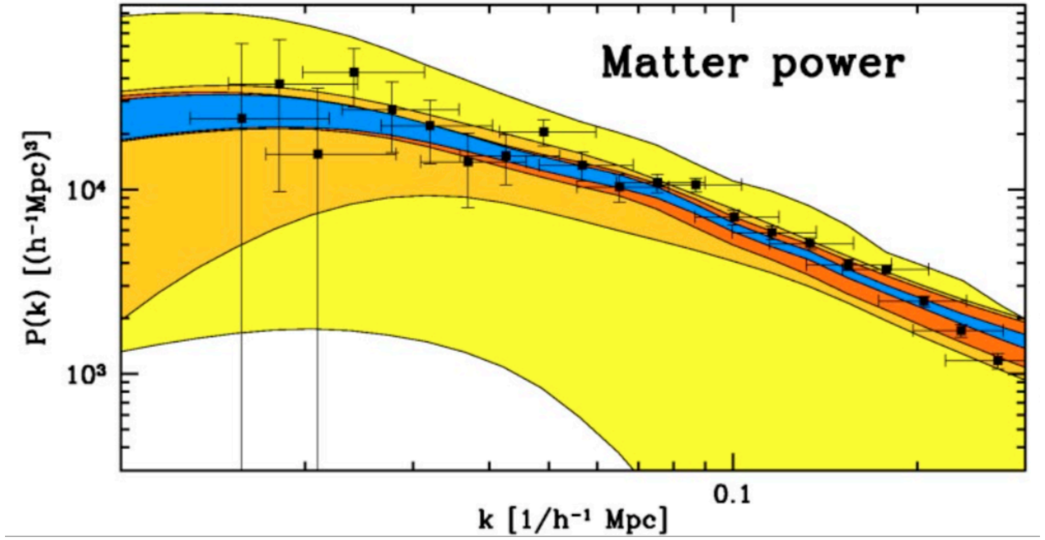


Evolution Equation

- Gauge Choice: $N = 1$(geodesic slicing), $\beta^a = 0$ (normal gauge)
- Einstein Equation
  - $\frac{\partial}{\partial t} \ln \Psi = \frac{1}{6} K$ $\qquad$ $\frac{\partial}{\partial t} \tilde{\gamma}_{ij} = 2\tilde{A}_{ij}$
  - $\frac{\partial}{\partial t} K = -2\tilde{A}_{ij}\tilde{A}^{ij} - \frac{1}{3} K^2 - 4\pi\rho$
  - $\frac{\partial}{\partial t} \tilde{A}_{ij} = K\tilde{A}_{ij} + 2\tilde{A}_{ik}\tilde{A}^k{}_j + \Psi^{-4}\left(\tilde{R}_{ij} - \frac{1}{3}\tilde{R}\tilde{\gamma}_{ij}\right)$
- Energy-Stress Conservation
  - $U_i = 0$ and $\frac{\partial}{\partial t}\rho = -\rho K$ $\qquad$ if $u^a = n^a$ initially

Initial Data on Σ_0

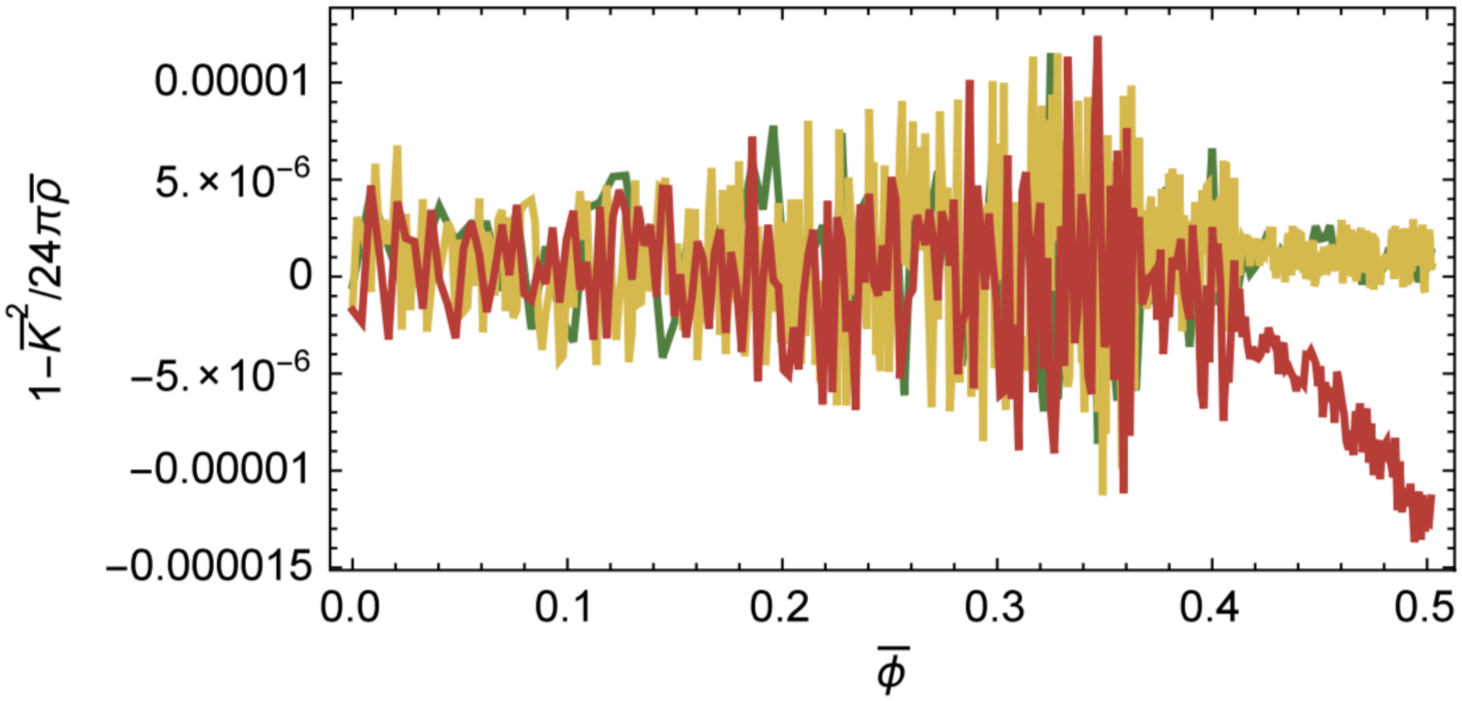
- Conformal Metric
  - $\tilde{\gamma}_{ij} = \delta_{ij}$: conformally flat assumption
- Extrinsic Curvature
  - K : constant
  - $A_{ij} = 0$
- Matter Density Power Spectrum
  - $P_k = \frac{4P_*}{3} \frac{k/k_*}{1+\frac{1}{3}(k/k_*)^4}$
- Conformal Factor
  - $\tilde{D}^i \tilde{D}_i \Psi = -2\pi\Psi^5 \left(\rho - \frac{1}{24\pi} K^2\right)$ [Hamiltonian Constraint]



Galaxy power from SDSS, PRD 69 103501

Behavior of Average

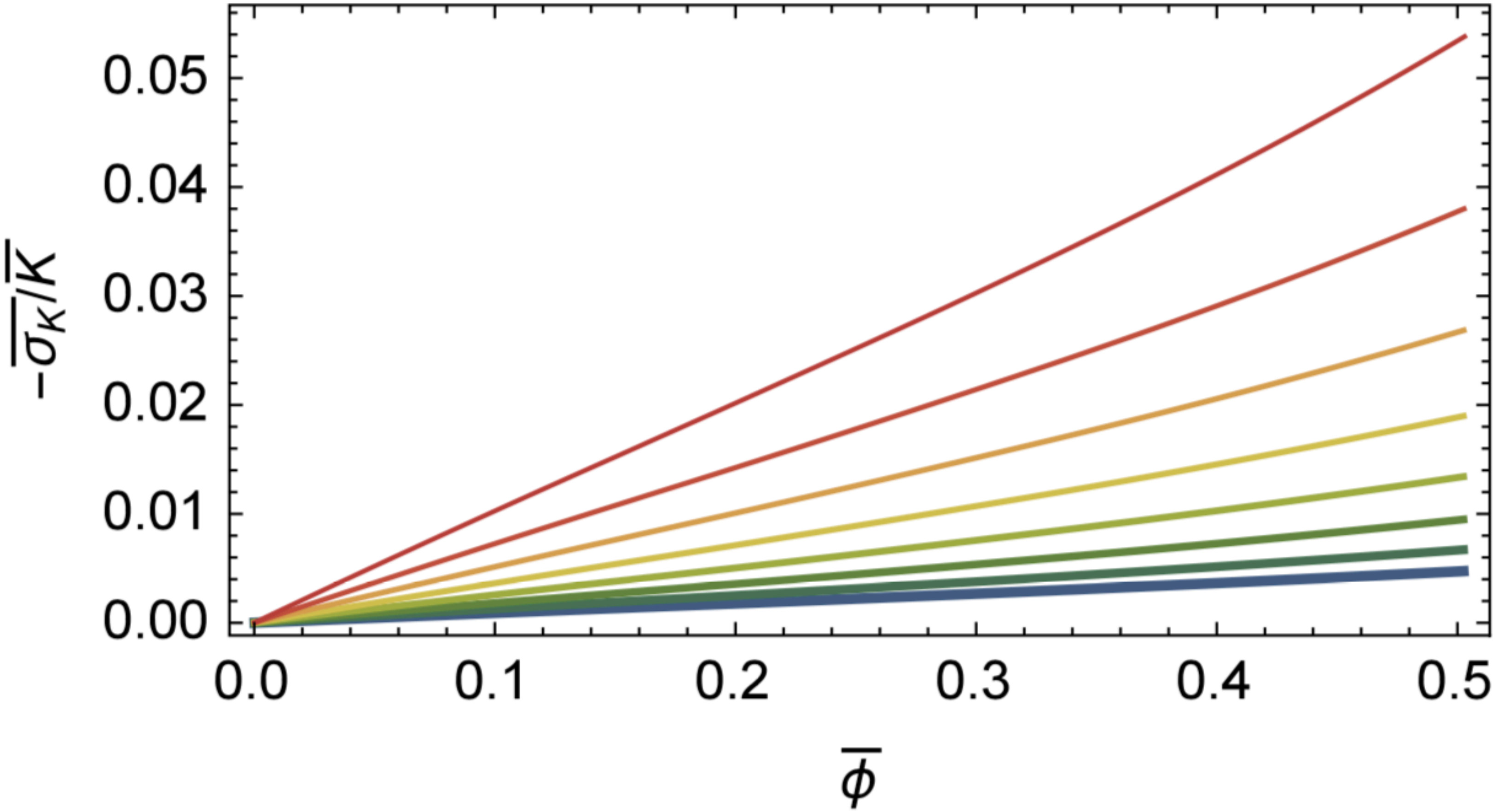
- $\phi = \ln \Psi = \frac{1}{2} \ln a$: e-folding / 2
- $\bar{\phi}$: spatial average of ϕ (proxy for time)
- $1 - \bar{K}^2/24\pi\bar{\rho}$: deviation of averaged behavior from FLRW



$\sigma_\rho/\bar{\rho} = 5\%$: initial inhomogeneity
Red : 64^3 grid
Green : 128^3 grid
Yellow : 256^3 grid

Growth of Inhomogeneity

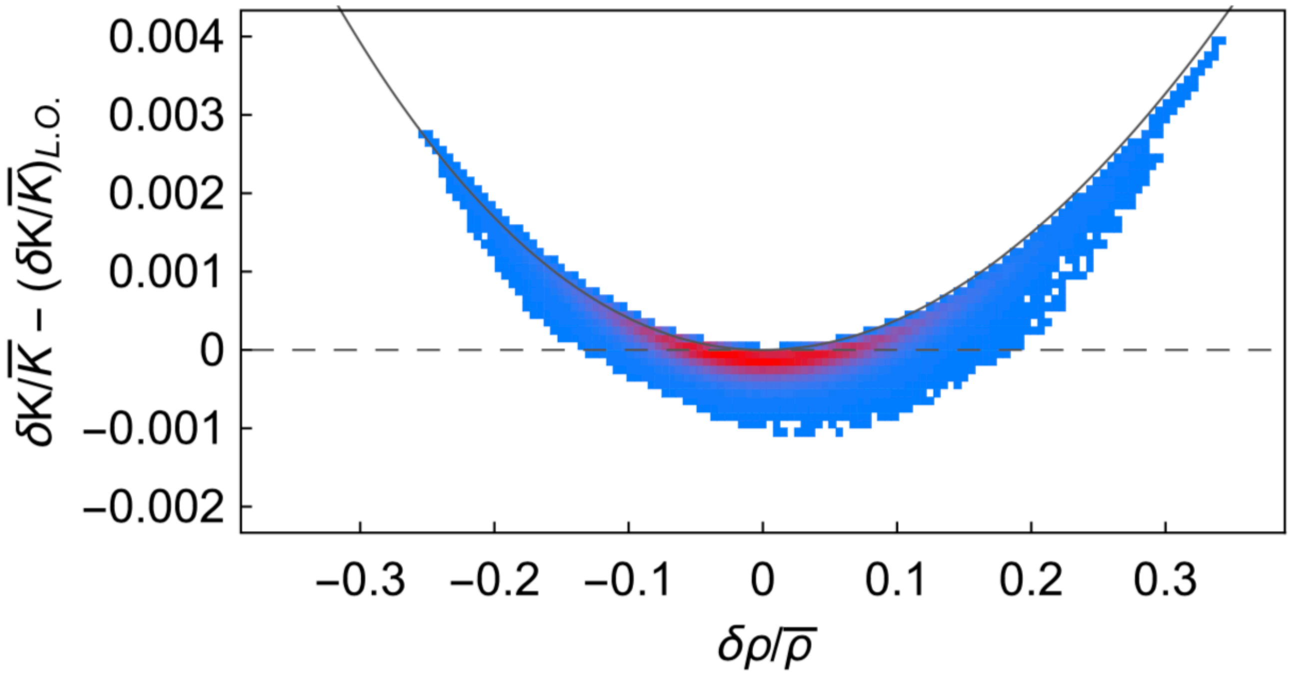
- $\sigma_K/\bar{K}$: variations of extrinsic curvature



$\sigma_\rho/\bar{\rho} = 0.9\% \sim 10.7\%$
(from blue to red)

Distribution of Deviation

- Linear perturbation in synchronous gauge
  - $\partial_t \delta\rho = \bar{\rho}\delta K + \bar{K}\delta\rho \qquad \partial_t \delta K = \frac{2}{3}\bar{K}\delta K + 4\pi\delta\rho$
- $\delta K/\bar{K} - (\delta K/\bar{K})_{L.O.}$ vs $\delta\rho/\bar{\rho}$



$\sigma_\rho/\bar{\rho} = 3.8\%$
At $\bar{\phi} = 0.5$
Dashed line : solution of linear perturbation
Blue : few points
Red : many points

Summary

- Understanding non-linear effects of GR on cosmology are getting more important.
- Numerical relativity has performed full GR simulation of spacetime successfully.
- Recently, numerical relativity has been applied to evolution of cosmological inhomogeneities.
- It indicates that the effect of non-linear inhomogeneities may be significant.

Constraint Violation

- $\mathcal{H} = \widetilde{D}^i \widetilde{D}_i \Psi - \frac{1}{8} \Psi \tilde{R} + \frac{1}{8} \Psi^5 \tilde{A}_{ij} \tilde{A}^{ij} - \frac{1}{12} \Psi^5 K^2 + 2\pi \Psi^5 \rho$
- $\mathcal{M}^i = \widetilde{D}_j \left(\Psi^6 \tilde{A}^{ij} \right) - \frac{2}{3} \Psi^6 \widetilde{D}^i K - 8\pi \Psi^{10} p^i$